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We present the first computation of the cosmological perturbations generated during inflation up to second-order in deviations from the homogeneous background solution. Our results, which fully account for the inflaton self-interactions as well as for the second-order fluctuations of the background metric, provide the exact expression for the gauge-invariant curvature perturbation bispectrum produced during inflation in terms of the slow-roll parameters or, alternatively, in terms of the scalar spectral n_S and and the tensor to adiabatic scalar amplitude ratio r. The bispectrum represents a specific non-Gaussian signature of fluctuations generated by quantum oscillations during slow-roll inflation. Our findings indicate that – for a broad class of single-field models of inflation – the level of non-Gaussianity in the cosmic microwave background anisotropies is large enough to be detectable by present and forthcoming satellite missions such as MAP and *Planck*.

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1. Introduction. Models of inflation based on the slow-roll of a single scalar field, the *inflaton*, provide the simplest causal mechanism for the generation of cosmological perturbations [1]. These fluctuations will then grow by gravitational instability to seed structure formation in the Universe, and to produce Cosmic Microwave Background (CMB) anisotropies. The generation and evolution of fluctuations during inflation has always been studied within linear theory, an approximation which is largely justified by their small amplitude at early times. On the other hand, there exist physical observables, such as the three-point function of scalar perturbations, or its Fourier tranform, the bispectrum, for which a perturbative treatment up to second-order is required, in order to obtain a self-consistent result.

The importance of the bispectrum comes from the fact that it represents the lowest order statistics able to distinguish non-Gaussian from Gaussian perturbations for which odd-order correlation functions necessarily vanish. An accurate calculation of the primordial bispectrum of cosmological perturbations has become an extremely important issue, as a number of present and future experiments, such as MAP and *Planck*, will allow to constrain or detect non-Gaussianity of CMB anisotropy data with high precision.

So far, the problem of calculating the bispectrum of perturbations produced during inflation has been addressed by either looking at the effect of inflaton self-interactions (which necessarily generate non-linearities in its quantum fluctuations) in a fixed de Sitter background [2], or by using the so-called stochastic approach to inflation [3], where back-reaction effects of field fluctuations on the background metric are partially taken into account. An intriguing result of the stochastic approach is that the dominant source of non-Gaussianity actually comes from non-linear gravitational perturbations, rather than by inflaton self-interactions.

In this Letter we provide for the first time the computation of the scalar perturbations produced during single-field slow-roll inflation up to second-order in deviations from the homogeneous background. We achieve different goals. First, we provide a gauge-invariant definition of the comoving curvature $\mathcal{R}$ at the second-order of perturbation theory. Secondly, we obtain an equation for the comoving curvature which shows that – contrary to the result obtained at the first-order – at the second-order the comoving curvature is not conserved on superhorizon scales, but is sourced by non-linear effects. Third, we obtain the exact expression for the gauge-invariant gravitational potential bispectrum during inflation, in terms of slow-roll parameters or, equivalently, of the spectral indices of the scalar and tensor power spectra. A detailed version of our findings will be presented in Ref. [4].

2. The second-order gauge-invariant comoving curvature perturbation. The primordial adiabatic density perturbation is associated with a perturbation of the spatial curvature. We consider the gauge-independent curvature perturbation $\mathcal{R}$ on slices orthogonal to comoving worldlines, which is related to the gauge-dependent curvature perturbation ψ on a generic slicing and to the inflaton perturbation $\delta\varphi$ in that gauge. At the first-order, the comoving curvature perturbation is given by the gauge-invariant formula (in conformal time τ) [5].

$$\mathcal{R}^{(1)} = \psi^{(1)} + \frac{\mathcal{H}}{\varphi_0'}\,\delta^{(1)}\varphi\,, \tag{1}$$

where φ_0 is the background value of the classical field, $\mathcal{H} = a'/a$ is the Hubble rate and $\psi^{(1)}$ and $\delta^{(1)}\varphi$ are the first-order curvature and inflaton perturbations, respectively.

Let us now consider the second-order perturbation. We expand the curvature perturbation ψ and the inflaton perturbation $\delta\varphi$ as $\psi = \psi^{(1)} + \frac{1}{2}\psi^{(2)}$ and $\delta\varphi = \delta^{(1)}\varphi + \frac{1}{2}\delta^{(2)}\varphi$. A second-order shift in the time-coordinate

$$\tau \to \tau + \xi^0_{(1)} + \frac{1}{2}\left(\xi^{0\prime}_{(1)}\xi^0_{(1)} + \xi^0_{(2)}\right) \tag{2}$$

induces the following gauge transformations [6,7]

$$\begin{aligned}\tilde{\psi}^{(2)} &= \psi^{(2)} + 2\xi^0_{(1)}\left(\psi^{(1)'} + 2\mathcal{H}\psi^{(1)}\right) \\ &\quad - \left(\mathcal{H}' + 2\mathcal{H}^2\right)\left(\xi^0_{(1)}\right)^2 - \mathcal{H}\xi^0_{(1)}{}'\xi^0_{(1)} - \mathcal{H}\xi^0_{(2)}\,,\end{aligned} \tag{3}$$

$$\begin{aligned}\delta^{(2)}\tilde{\varphi} &= \delta^{(2)}\varphi + \xi^0_{(1)}\left(\varphi_0''\xi^0_{(1)} + \varphi_0'\xi^0_{(1)}{}' + 2\delta^{(1)}\varphi'\right) \\ &\quad + \varphi_0'\xi^0_{(2)}\,.\end{aligned} \tag{4}$$

We find that the second-order gauge invariant comoving curvature perturbation $\mathcal{R} = \mathcal{R}^{(1)} + \frac{1}{2}\mathcal{R}^{(2)}$ is provided by* (for further details see [4])

$$\begin{aligned}\mathcal{R} &= \mathcal{R}^{(1)} + \frac{1}{2}\left[\mathcal{H}\frac{\delta^{(2)}\varphi}{\varphi_0'} + \psi^{(2)}\right] \\ &\quad + \frac{1}{2}\frac{(\psi^{(1)\prime} + 2\mathcal{H}\psi^{(1)} + \mathcal{H}\delta^{(1)}\varphi'/\varphi_0')^2}{\mathcal{H}' + 2\mathcal{H}^2 - \mathcal{H}\,\varphi_0''/\varphi_0'}\,.\end{aligned} \tag{5}$$

3. The computation. We can now choose to work with a generalized longitudinal gauge, where the space-time metric can be written as

$$\begin{aligned}g_{00} &= -a^2(\tau)(1 + 2\phi^{(1)} + \phi^{(2)})\,, \\ g_{0i} &= 0\,, \\ g_{ij} &= a^2(\tau)\left[(1 - 2\psi^{(1)} - \psi^{(2)})\delta_{ij}\right. \\ &\quad \left. + \frac{1}{2}\left((\partial_i\chi^{(2)}_j + \partial_j\chi^{(2)}_i + \chi^{(2)}_{ij})\right)\right]\,,\end{aligned} \tag{6}$$

where $\chi^{(2)}_i$ $(i = 1,2,3)$ is a transverse (*i.e.* divergence-free) vector and $\chi^{(2)}_{ij}$ is a transverse trace-free tensor with respect to the 3-dimensional space with metric δ_{ij}. When considering perturbations up to first-order only the vector parts are zero and the tensor modes are neglected as they can be easily shown to give a negligible contribution to the bispectrum. Instead, in the second-order theory the vector and the tensor contributions are generated by the scalar perturbations even if they are initially zero [7].

In the following, we briefly summarize the procedure to obtain the equation of motion for $\mathcal{R}^{(2)}$. For further details, see Ref. [4]. We first calculate the perturbed Einstein equations $\delta^{(2)}G^\mu{}_\nu = \kappa^2\delta^{(2)}T^\mu{}_\nu$, where $\kappa^2 = 8\pi G_N$ and $T^\mu{}_\nu$ is the energy-momentum tensor of the inflaton field. We use the $(0-0)$-component of Einstein equations, the divergence of the $(0-i)$-component and the trace of the $(i-j)$-component, both performed with the background metric δ_{ij}. The divergence and the trace operations make the vector and the tensor modes disappear from the equations. Thus, we are left with three equations in the three unknown functions, $\phi^{(2)}, \psi^{(2)}$, and $\delta^{(2)}\varphi$. From the divergence of the $(0-i)$-component of Einstein equations it is possible to recover an expression for $\delta^{(2)}\varphi$ [4]

$$\frac{1}{2}\delta^{(2)}\varphi = \frac{\left(\psi^{(2)\prime} + \mathcal{H}\,\phi^{(2)} + \triangle^{-1}\alpha\right)}{\kappa^2\,\varphi_0'} - \frac{\triangle^{-1}\beta}{\varphi_0'}\,, \tag{7}$$

where

$$\begin{aligned}\alpha &= 2\psi^{(1)\prime}\partial_i\,\partial^i\psi^{(1)} + 10\,\partial_i\psi^{(1)\prime}\partial^i\psi^{(1)} \\ &\quad + 8\psi^{(1)}\,\partial_i\,\partial^i\psi^{(1)\prime}\,,\end{aligned} \tag{8}$$

$$\begin{aligned}\beta &= \partial_i\,\partial^i\delta^{(1)}\varphi\,\delta^{(1)}\varphi' + \partial^i\delta^{(1)}\varphi\,\partial_i\delta^{(1)}\varphi' \\ &\quad + 2\,\psi^{(1)}\,\partial_i\,\partial^i\delta^{(1)}\varphi\,\varphi_0' + 2\,\partial_i\psi^{(1)}\,\partial^i\delta^{(1)}\varphi\,\varphi_0'\,,\end{aligned} \tag{9}$$

and $\triangle^{-1}$ is the inverse of the Laplacian operator for the three spatial-coordinates. The expression (7) can be used in the trace of the $(i-j)$ equation to obtain [4]

$$\psi^{(2)} = \phi^{(2)} - \triangle^{-1}\gamma, \tag{10}$$

where

$$\begin{aligned}\gamma &= 24\,\frac{a''}{a}\,\psi^{(1)^2} - 12\left(\frac{a'}{a}\right)^2\psi^{(1)^2} + 24\frac{a'}{a}\,\psi^{(1)}\psi^{(1)\prime} \\ &\quad + 7\,\partial_i\psi^{(1)}\,\partial^i\psi^{(1)} + 8\,\partial_i\,\partial^i\psi^{(1)}\,\psi^{(1)} + 3\psi^{(1)\prime 2} \\ &\quad + 3\,\triangle^{-1}\alpha' + 6\,\frac{a'}{a}\,\triangle^{-1}\alpha - 3\,\kappa^2\frac{a'}{a}\,\triangle^{-1}\beta' \\ &\quad - 6\frac{a'}{a}\,\kappa^2\,\triangle^{-1}\beta + 3\kappa^2\left[\frac{1}{2}\,(\delta^{(1)}\varphi')^2 - \frac{1}{6}\partial_i\delta^{(1)}\varphi\partial^i\delta^{(1)}\varphi\right. \\ &\quad \left. + 2\psi^{(1)^2}\varphi_0'^2 - \frac{1}{2}(\delta^{(1)}\varphi)^2\frac{\partial^2 V}{\partial\varphi^2}\,a^2 - 2\psi^{(1)}\delta^{(1)}\varphi'\varphi_0'\right]. \end{aligned} \tag{11}$$

Here $V(\varphi)$ is the inflaton potential. From Eq. (7) we finally obtain

$$\begin{aligned}\frac{1}{2}\delta^{(2)}\varphi &= \frac{\left(\phi^{(2)\prime} + \mathcal{H}\,\phi^{(2)} + \triangle^{-1}\alpha\right)}{\kappa^2\,\varphi_0'} \\ &\quad - \frac{\triangle^{-1}\beta}{\varphi_0'} - \frac{\triangle^{-1}\gamma'}{\kappa^2\,\varphi_0'}\,.\end{aligned} \tag{12}$$

Plugging Eq. (10) into the $(0-0)$ Einstein equation, an equation of motion for $\phi^{(2)}$ is derived

$$\phi^{(2)\prime\prime} - \partial_i\partial^i\phi^{(2)} + 2\left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\phi^{(2)\prime} + 2\left(\mathcal{H}' - \frac{\varphi_0''}{\varphi_0'}\mathcal{H}\right)\phi^{(2)}$$

$$= 12\mathcal{H}^2\left(\psi^{(1)}\right)^2 + 3\left(\psi^{(1)\prime}\right)^2 + 8\psi^{(1)}\partial_i\partial^i\psi^{(1)} + 3\partial_i\psi^{(1)}\partial^i\psi^{(1)}$$

*Analogously, one can find a second-order gauge invariant expression for the curvature perturbation on uniform-density hypersurfaces ζ [4].

$$-\triangle^{-1}\alpha'+2\frac{\varphi_0''}{\varphi_0'}\triangle^{-1}\alpha-2\frac{\varphi_0''}{\varphi_0'}\triangle^{-1}\gamma'+\triangle^{-1}\gamma''-2\kappa^2\frac{\varphi_0''}{\varphi_0'}\triangle^{-1}\beta$$

$$+\kappa^2\triangle^{-1}\beta'+2\mathcal{H}\triangle^{-1}\alpha+\mathcal{H}\triangle^{-1}\gamma'-2\kappa^2\mathcal{H}\triangle^{-1}\beta$$

$$-\gamma+\kappa^2\Big[-\frac{1}{2}(\delta^{(1)}\varphi')^2-\frac{1}{2}\partial_i\delta^{(1)}\varphi\partial^i\delta^{(1)}\varphi-2{\psi^{(1)}}^2\varphi_0'^2$$

$$-\frac{1}{2}(\delta^{(1)}\varphi)^2\frac{\partial^2 V}{\partial\varphi^2}a^2+2\psi^{(1)}\delta^{(1)}\varphi'\varphi_0'\Big]\,. \qquad (13)$$

Notice that this equation is exact at any order in the expansion in terms of the slow-roll parameters. Solving this equation in the long-wavelength limit and expanding it to first-order in the slow-roll parameters†, we find the following equation for $\mathcal{R}^{(2)}$ in cosmic time t (a dot indicates differentation with respect to t) [4]

$$\frac{1}{2}H\dot{\mathcal{R}}^{(2)}=\mathcal{K}\left[\left(\mathcal{R}^{(1)}\right)^2\right]\,, \qquad (14)$$

with

$$\mathcal{K}\left[\left(\mathcal{R}^{(1)}\right)^2\right]=(-9\epsilon+2\eta)\,H^2\left(\mathcal{R}^{(1)}\right)^2\,. \qquad (15)$$

Here $\epsilon=-\dot{H}/H^2$ and $\eta=\epsilon-(\ddot{\varphi}_0/H\dot{\varphi}_0)$ are the usual slow-roll parameters [1]. From Eq. (14) we conclude that at the second-order perturbation level and contrary to what happens for the first-order comoving curvature $\mathcal{R}^{(1)}$, the comoving curvature is not conserved on superhorizon scales. It is indeed sourced by the non-linear term $\left(\mathcal{R}^{(1)}\right)^2$.

The expression for the second-order comoving curvature perturbation, calculated at the end of inflation t_f, reads

$$\frac{1}{2}\mathcal{R}^{(2)}(t_f)=(-9\epsilon+2\eta)\,(\mathcal{R}^{(1)})^2 H\,(t_f-t_H(k))\,, \qquad (16)$$

where $t_H(k)$ is the time at which the wavenumber k crosses the Hubble radius during inflation. The overall comoving curvature perturbation $\mathcal{R}=\mathcal{R}^{(1)}+\mathcal{R}^{(2)}/2$ represents the initial condition for the post-inflationary evolution of the cosmological perturbations.

4. Non-Gaussianity of the curvature perturbation. The comoving curvature perturbation receives at the second-order of perturbation a contribution which is quadratic in $\mathcal{R}^{(1)}$. If the quadratic term is not negligible, the total curvature perturbation will have a non-Gaussian (χ^2) component. We can readily recast Eq. (16) in a form which clearly reveals the level of non-Gaussianity implied by our second-order inflationary calculation. Reminding that the Bardeen gauge-invariant potential Φ [8] and the curvature perturbation $\mathcal{R}$ are related by $\Phi=\frac{3}{5}\mathcal{R}$, the following simple relation in configuration space holds

$$\Phi=\Phi_{\rm Gauss}+f_{\rm NL}\left(\Phi^2_{\rm Gauss}-\langle\Phi^2_{\rm Gauss}\rangle\right)\,, \qquad (17)$$

which is valid on superhorizon scales. Here $\Phi_{\rm Gauss}=\frac{3}{5}\mathcal{R}^{(1)}$ is a Gaussian random field. According to Eq. (16), the non-Gaussianity (or non-linearity) parameter $f_{\rm NL}$ is given by

$$f_{\rm NL}=\frac{5}{3}\,(-9\epsilon+2\eta)\,\Delta N\,, \qquad (18)$$

where ΔN is the number of e-foldings from the time at which a given scale crosses the horizon and the end of inflation; for large-scale CMB anisotropies, $\Delta N\approx 60$. Comparison with previous calculations shows two main differences. First, the expression in terms of the slow-roll parameters ϵ and η differs from the one in Refs. [3,9]. This is because in the stochastic inflation approach of Ref. [3] second-order perturbations of the metric were oversimplified as a local modification of the Hubble expansion rate during inflation. Second and most noticeably, our expression contains an extra factor ΔN. Its origin is manifest from Eq. (16). The second-order comoving curvature perturbation is fuelled on superhorizon scales by a constant source and its value at the end of inflation is proportional to the time the perturbation spends outside the horizon.

It is interesting to express the above result in terms of the spectral indices n_S, for the scalar perturbations, and n_T, for the gravitational waves. These are given by the two relations $n_S-1=-6\epsilon+2\eta$ and $n_T=-2\epsilon$, which apply to single-field slow-roll inflationary models. We then get

$$f_{\rm NL}=50\,[2(n_S-1)+3n_T]\left(\frac{\Delta N}{60}\right)$$
$$=50\,[2(n_S-1)-6r]\left(\frac{\Delta N}{60}\right)\,. \qquad (19)$$

In the last passage we have made use of the consistency relation $r=-n_T/2$ relating the tensor spectral index n_T and the tensor to adiabatic scalar amplitude ratio r. The latter is defined as $r=A_T^2/A_{\mathcal{R}}^2$, where $A_{\mathcal{R}}^2=(4/25)\mathcal{P}_{\mathcal{R}}$ and $A_T^2=(1/100)\mathcal{P}_T$ are the amplitudes of the scalar and tensor perturbation spectra, respectively [1].

The primordial gauge-invariant potential bispectrum leads to a nonzero CMB bispectrum via the Sachs-Wolfe effect $(\Delta T/T)_{SW}=(1/3)\Phi$. Using the notation of Ref. [10] the reduced bispectrum can be expressed in terms of angular power spectra C_ℓ as

$$b_{\ell_1\ell_2\ell_3}=6f_{\rm NL}\left(C_{\ell_1}C_{\ell_2}+C_{\ell_1}C_{\ell_3}+C_{\ell_2}C_{\ell_3}\right)\,. \qquad (20)$$

†Notice that the contribution of the inflaton self-interaction are next-to-leading in the slow roll parameters and therefore can be safely neglected.

5. Discussion and Conclusions. Eqs. (16) and (19) represent the main result of our Letter. For the typical values of n_S and n_T predicted by many slow-roll inflationary models, $f_{\rm NL}$ can easily reach large values. For example, chaotic inflationary models with monomial and exponential potentials [1,11], $r \simeq 5\,(1-n_S)$ and $f_{\rm NL} \sim 10^3\,(n_S-1)$. Even tiny deviations from a scale-invariant spectrum make the primordial non-Gaussianity non-negligible. To give a flavour of the magnitude of $f_{\rm NL}$, a 5% tilt of the scalar primordial spectrum – well allowed by present CMB data [12,13] – implies $f_{\rm NL} \sim 50$. The possible presence of non-Gaussianity in primordial cosmological perturbations is only mildly constrained by existing observations [14,15]. Recent analyses of the angular bispectrum from 4-year COBE data [16] yield a weak upper limit, $|f_{\rm NL}| < 1.5 \times 10^3$. The analysis of the diagonal angular bispectrum of the Maxima dataset [17] also provides a very weak constraint: $|f_{\rm NL}| < 2330$. According to Ref. [10], however, the minimum value of $|f_{\rm NL}|$ that will become detectable from the analysis of MAP and *Planck* data, after properly subtracting detector noise and foreground contamination, is about 20 and 5, respectively. These results imply that the level of non-Gaussianity emerging from our second-order calculation is detectable by MAP and *Planck* for a broad class of single-field slow-roll inflationary models.

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